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A Modification of Inclusion of a Zero of a Function using Interval Method

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ABSTRACT

Interval method is used for the inclusion of a zero of a function. The Ehrman(EHR) method considers the Newton's iteration in finding the root of a function. This method is modified by using the mid point in the procedure and improved method has a faster convergence rate and less processing time. In this paper, the convergence analysis and the numerical results are shown.

Keywords: Interval analysis, zero of a function, inclusion, CPU time, rate of convergence

INTRODUCTION

Aberth [1], Alefeld and Herzberger [2], Braess and Hadeler [4] and Ehrlic [7], have carried out research on iterative procedures in finding zero of a function. Gargantini [9] and Hansen [10] have shown that iterative procedure which involved interval analysis approach gives better results and the inclusion of a zero is always guaranteed. EHR method which was discussed in Alefeld and Herzberger [3] is based on the work of Ehrmann [6] have proved the order of convergence to be $p+1$, $(p\geq 1)$.

In the paper, a modified EHR method called the MEHR method where the mid point approach is used in the iterative procedure is discussed. The MEHR method gives better result and shortens the number of iteration. Furthermore, the order of convergence of this method is at least $p+3$, $(p\geq 1)$.

BACKGROUND OF THE PROBLEM

The EHR method is about the inclusion of a zero of a function in the last smallest interval of the sequence of intervals $\left\{X^{(k)}\right\}_0^{\infty}$. Alefeld and Herzberger [3], Caparani and Madesan [5] and Ely [8] have widely discussed about this method. Consider the following which is actually generated from the Taylor Series

$$0 = f(\xi) = f\left(x^{(k)}\right) + f'(x^{(k)})(\xi - x^{(k)}) + \ldots + \frac{1}{(i+1)!} f^{(i+1)}\left(x^{(k)}\right)\left(\xi - x^{(k)}\right)^{(i+1)} + \frac{1}{(i+2)!} f^{(i+2)}(\eta_{i+2})\left(\xi - x^{(k)}\right)^{(i+2)}. \quad (1)$$

Thus,

$$\xi = x^{(k)} - \frac{1}{f'\left(x^{(k)}\right)}\left[f\left(x^{(k)}\right) + \sum_{\upsilon=2}^{i+1} \frac{f^{(\upsilon)}\left(x^{(k)}\right)}{\upsilon!}\left(\xi - x^{(k)}\right)^{\upsilon} + \frac{f^{(i+2)}\left(\eta_{i+2}\right)}{(i+2)!}\left(\xi - x^{(k)}\right)^{(i+2)} \right] \quad (2)$$

where $\eta_{i+2} \in \left(x^{(k)}, \xi\right)$.

By using the concept of interval analysis in the equation (2), we have

$$X^{(k+1)} = \left\{ x^{(k)} - \frac{1}{f'\left(x^{(k)}\right)}\left[f\left(x^{(k)}\right) + \sum_{\upsilon=2}^{i} \frac{1}{\upsilon!} f^{(\upsilon)}(x^{(k)})(X^{(k)} - x^{(k)})^{\upsilon} + \frac{1}{(i+1)!} F_{i+1}(X^{(k)} - x^{(k)})^{(i+1)} \right] \right\} \cap X^{(k)} \quad (3)$$

for $1 \le i \le p$, $k \ge 0$ and $F_{i+1} = f^{(i+1)}\left(X^{(k)}\right)$.

The sets $\{X^{(k)}\}$ generated from equation (3) satisfy

$$X^{(0)} \supset X^{(1)} \supset X^{(2)} \supset \ldots \supset X^{(k)} \supset X^{(k+1)} \ldots$$

$$\xi \in X^{(k)}, \forall k \geq 0$$

and

$$\lim_{k \to \infty} X^{(k)} = \xi = [\xi, \xi] .$$

The point $x \in X^{(k)}$ can be chosen randomly, but in this paper x is taken as the mid point of $X^{(k)}$ and written as $x = m(X^{(k)})$.

The interval $M = [m_1, m_2]$ must satisfy

$$0 < m_1 \leq \frac{f(x) - f(\xi)}{x - \xi} = \frac{f(x)}{x - \xi} \leq m_2 < \infty , \xi \neq x \in X^{(k)}, (\forall k \geq 0).$$

CONVERGENCE THEOREM

The following theorem is the basic concept of convergence used in this work.

Theorem 1:

Let I be an iteration procedure with the limit x^*, and let $\wp(I, x^*)$ be the set of all sequences $\{X^{(k)}\}$ generated by I having the properties that

$$\lim_{k \to \infty} X^k = x^* \text{ and } x^* \subseteq x^{(k)}, k \geq 0 .$$

If there exists a $p \geq 1$ and a constant γ such that for all $\{X^{(k)}\} \in \wp(I, x^*)$ and for a norm $\| \circ \|$, it holds that $\| x^{(k+1)} \| \leq \gamma \| x^{(k)} \|^{p}$

then it follows that the R-order of I satisfies,

$$O_R\left(I,x^*\right) \geq p.$$

and the R-order of convergence of I is at least p.

The prove of this theorem is available in [3].

EHR METHOD

Theorem 2:

Let f be a $(p+1)$–times continuously differentiable function, $p \geq 1$ and let $f\left(x_1^{(0)}\right) < 0$ and $f\left(x_2^{(0)}\right) > 0$ be valid in $X^{(0)} = [x_1^{(0)}, x_2^{(0)}]$. Let ξ be the zero of f in $X^{(0)}$ and let the interval $M = \left[m_1, m_2\right]$ be defined by

$$0 < m_1 \leq \frac{f(x) - f(\xi)}{x - \xi} = \frac{f(x)}{x - \xi} \leq m_2 < \infty \ , \xi \neq x \in X^{(0)}.$$

Furthermore, let $f^{(i)}(x) \in F_i$ be the ith derivative of f, $x \in X^{(0)}$ where F_i is the interval evaluation of the derivative $f^{(i)}$ such that $F_i \supseteq f^{(i)}\left(X^{(k)}\right)$ over $X^{(k)}$.

We now consider the following iteration:

$$x^{(k)} = m\left(X^{(k)}\right) \in X^{(k)}$$

$$X^{(k+1,0)} = \left\{x^{(k)} - \frac{f\left(x^{(k)}\right)}{M}\right\} \cap X^{(k)}.$$

$$X^{(k+1,i)} = \left\{x^{(k)} - \frac{1}{f'\left(x^{(k)}\right)}\left[f\left(x^{(k)}\right)\right.\right.$$

$$\left. + \sum_{\upsilon=2}^{i} \frac{f^{(\upsilon)}\, x^{(k)}}{\upsilon!}\left(X^{(k+1,i-1)} - x^{(k)}\right)^{\upsilon}\right.$$

$$+\frac{1}{(i+1)!}F_{I+1}\left(X^{(k+1,i-1)}-x^{(k)}\right)^{(i+1)}\Bigg]\Bigg\}\cap X^{(k+1,i-1)},$$

$$i = 1,2,\ \ldots,\ p$$

$$X^{(k+1)} = X^{(k+1,p)}.$$

Hence

$$\xi \in X^{(k)}, k \geq 0$$

$$X^{(0)} \supset X^{(1)} \supset X^{(2)} \supset \ldots$$

$$\lim_{k\to\infty} X^{k} = \xi,$$

and

$$d\left(X^{(k+1)}\right) \leq \gamma\left(d\left(X^{(k)}\right)\right)^{(p+1)} \quad (\gamma \geq 0).$$

Therefore, the order of convergence of EHR is at least p+1, $(p \geq 1)$.
This theorem was proved in [3].

MEHR METHOD

By using the same hypothesis given in EHR method, MEHR method uses the midpoint $x^{(k+1,i-1)}$, $(i = 1,\ldots,p)$ to substitute $x^{(k)}$.

Theorem 3:

Let f be a $(p+1)$–times continuously differentiable function, let $f\left(x_1^{(0)}\right) < 0$ and $f\left(x_2^{(0)}\right) > 0$ be valid in $X^{(0)} = [x_1^{(0)}, x_2^{(0)}]$. Let ξ be the zero of f in $X^{(0)}$ and let the interval $M = [m_1, m_2]$ be defined by $0 < m_1 \leq \dfrac{f(x) - f(\xi)}{x - \xi} = \dfrac{f(x)}{x - \xi} \leq m_2 < \infty$, $\xi \neq x \in X^{(0)}$. Furthermore, let $f^{(i)}(x) \in F_i$ be the i^{th} derivative of f, $x \in X^{(0)}$ where F_i is the interval evaluation of the derivative $f^{(i)}$ such that $F_i \supseteq f^{(i)}\left(X^{(k)}\right)$ over $X^{(k)}$.

We now consider the following iteration:

$$x^{(k)} = m\left(X^{(k)}\right) \in X^{(k)} .$$

$$X^{(k+1,0)} = \left\{ x^{(k)} - \frac{f\left(x^{(k)}\right)}{M} \right\} \cap X^{(k)} .$$

For $i = 1,\ldots,p$

$$x^{(k+1,i-1)} = m\left(X^{(k+1,i-1)}\right) \in X^{(k+1,i-1)} .$$

$$X^{(k+1,i))} = \left\{ x^{(k+1,i-1)} - \frac{1}{f'\left(x^{(k+1,i-1)}\right)}\left[f\left(x^{(k+1,i-1)}\right) + \sum_{\upsilon=2}^{i} \frac{1}{\upsilon!} f^{(\upsilon)}\left(x^{(k+1,i-1)}\right)\left(X^{(k+1,i-1)} - x^{(k+1,i-1))}\right)^{\upsilon} + \frac{1}{(i+1)!} F_{i+1}\left(X^{(k+1,i-1)} - x^{(k+1,i-1)}\right)^{(i+1)} \right] \right\}$$

$$\cap X^{(k+1,i-1)} .$$

$$X^{(k+1,p)} = X^{(k+1)},\ k \geq 0 .$$

Hence

i. $\xi \in X^{(k)}, k \geq 0$
ii. $X^{(0)} \supset X^{(1)} \supset X^{(2)} \supset \ldots$
iii. $\lim_{k\to\infty} X^{k} = \xi$,

and

$$\text{iv. } d\left(X^{(k+1)}\right) \leq \gamma\left(d\left(X^{(k)}\right)\right)^{(p+3)}, (\gamma \geq 0).$$

Therefore, the order of convergence of MEHR is at least p+$3, (p \geq 1)$.

Proof:

(i) : Assume that $\xi \in X^{(k)}$, for some $k \geq 0$. In [3] it has been proved that $\xi \in X^{(k+1,0)}$.

We want to prove that $\xi \in X^{(k+1,i)}$ for all *i*.

Now,

$$(\xi - x^{(k)}) \in X^{(k+1,i)} - x^{(k)}$$

and
$$(\xi - x^{(k+1,i-1)}) \in X^{(k+1,i-1)} - x^{(k+1,i-1)}$$

From Taylor's formula , we get

$$0 = f(\xi) = f\left(x^{(k+1,i-1)}\right) + f'\left(x^{(k+1,i-1)}\right)\left(\xi - x^{(k+1,i-1)}\right)$$

$$+ \frac{f''\left(x^{(k+1,i-1)}\right)}{2!}\left(\xi - x^{(k+1,i-1)}\right)^2 + \dots$$

$$+ \frac{f^{(i+1)}\left(x^{(k+1,i-1)}\right)}{(i+1)!}\left(\xi - x^{(k+1,i-1)}\right)^{(i+1)}$$

$$+ \frac{f^{(i+2)}\left(\eta_{i+2}\right)}{(i+2)!}\left(\xi - x^{(k+1,i+1)}\right)^{(i+2)},$$

$$\left(\eta_{i+2} \in (\xi, x^{(k+1,i-1)}\right)$$

Finally we get the following result,

$$\xi = x^{(k+1,i-1)} - \frac{1}{f'\left(x^{(k+1,i-1)}\right)}\left[f\left(x^{(k+1,i-1)}\right) + \frac{f''\left(x^{(k+1,i-1)}\right)}{2!}\left(\xi - x^{(k+1,i-1)}\right)^2 + \frac{f^{(i+1)}\left(x^{(k+1,i-1)}\right)}{(i+1)!}\left(\xi - x^{(k+1,i-1)}\right)^{(i+1)} + \frac{f^{(i+2)}(\eta_{i+2})}{(i+2)!}\left(\xi - x^{(k+1,i-1)}\right)^{(i+2)}\right].$$

Therefore

$$\xi \in \left\{ x^{(k+1,i-1)} - \frac{1}{f'\left(x^{(k+1,i-1)}\right)}\left[f\left(x^{(k+1,i-1)}\right) + \sum_{\upsilon=2}^{i+1} \frac{f^{(\upsilon)}\left(x^{(k+1,i-1)}\right)}{\upsilon!}\left(X^{(k+1,i-1)} - x^{(k+1,i-1)}\right)^{(\upsilon)} + \frac{F_{i+2}}{(i+2)!}\left(X^{(k+1,i-1)} - x^{(k+1,i-1)}\right)^{(i+2)}\right]\right\} \cap X^{(k+1,i)} = X^{(k+1,i+1)}.$$

With $F_{i+2} = f^{(i+2)}(\eta_{i+2}), \xi \in X^{(k+1,i)} \quad (0 \le i \le p)$ and

$\xi \in X^{(k+1)} = X^{(k+1,p)}$, then $\xi \in X^{(k)}, k \ge 0$.

(ii): Using theorem 4 in [3], we have

$$d\left(X^{(k+1,0)}\right) \le \left(1 - \frac{m_1}{m_2}\right) d\left(X^{(k)}\right).$$

From the above procedure , $X^{(k+1)} = X^{(k+1,0)} \cap X^{(k)}$, then

$$d\left(X^{(k+1)}\right) \le \left(1-\frac{m_1}{m_2}\right) d\left(X^{(k)}\right),\ k \ge 0 .$$

Therefore $1-\frac{m_1}{m_2} < 1$, and $d\left(X^{(k+1)}\right) < d\left(X^{(k)}\right)$, $k \ge 0$.

Hence

$$X^{(0)} \supset X^{(1)} \supset X^{(2)} \supset \ldots$$

(iii): As in theorem 4 in [3] and the result from (ii) we conclude that $\lim_{k\to\infty} X^{(k)} = \xi$.

(iv): We have $d\left(X^{(k+1,0)}\right) \le d\left(X^{(k)}\right)$ $(k \ge 0)$, and that

$$d\left(X^{(k+1,1)}\right) \le d\left(X^{(k+1,0)}\right)$$

$$\le d \left(x^{(k+1,0)} - \frac{1}{f'\left(x^{(k+1,0)}\right)} \left[f\left(x^{(k+1,0)}\right) + \frac{F_2}{2!}\left(X^{(k+1,0)} - x^{(k+1,0)}\right)^2 \right] \right)$$

$$\le \frac{1}{2!}\left\{ d\left(\frac{F_2}{M}\right) \left[\left(X^{(k)} - X^{(k)}\right) + \left(x^{(k)} - x^{(k+1,0)}\right) \right]^4 \right\}$$

$$\le \frac{1}{2!}\left\{ d\left(\frac{F_2}{M}\right) \left| \left[\left(-d\left(X^{(k)}\right), d\left(X^{(k)}\right)\right) + \left(x^{(k)} - x^{(k+1,0)}\right) \right]^4 \right| \right.$$

$$\left. + \left|\frac{F_2}{M}\right| d\left[\left(-d\left(X^{(k)}\right), d\left(X^{(k)}\right)\right) + \left(x^{(k)} - x^{(k+1,0)}\right) \right]^4 \right\}.$$

$$= \frac{18}{2!}\left|\frac{F_2}{M}\right|\left[d\left(X^{(k)}\right)\right]^4$$

$$= \gamma_1\left[d\left(X^{(k)}\right)\right]^4$$

where $\gamma_1 = \frac{18}{2!}\left|\frac{F_2}{M}\right|$ independent of k.

Then

$$d\left(X^{(k+1,1)}\right) \le \gamma_1\left(d\left(X^{(k)}\right)\right)^4.$$

It can easily be shown that $d\left(X^{(k+1,i)}\right) \le \gamma_i\left(d\left(X^{(k)}\right)\right)^{(i+3)}$, for $i \ge 1$. Where γ_i is independent of k . This is also true for $1 \le i \le p$. For the last iteration ($i = p$), we get

$$d\left(X^{(k+1)}\right) = d\left(X^{(k+1,p)}\right) \le \gamma_p\left(d\left(X^{(k)}\right)\right)^{(p+3)}$$

then

$$d\left(X^{(k+1)}\right) \le \gamma\left(d\left(X^{(k)}\right)\right)^{(p+3)}, \left(\gamma = \gamma_p\right)$$

with γ_i is independent of k and the theorem is proved.

ALGORITHM MEHR

Initial interval $X^{(0)} = [x_1^{(0)}, x_2^{(0)}]$, $M = [m_1, m_2] = \left\{f'(x) \mid x \in X^{(0)}\right\}$, $0 < m_1 \le \frac{f(x) - f(\xi)}{x - \xi} \le m_2 < \infty$, $\xi \ne x \in X^{(0)}$ and $\varepsilon = 10^{-14}$.

1. Fix $k := 0$
2. $x^{(k)} = m(X^{(k)})$, (midpoint of $X^{(k)}$) ; $i := 0$

$$X^{(k+1,0)} = \left\{x^{(k)} - \frac{f\left(x^{(k)}\right)}{M}\right\} \cap X^{(k)}$$

3. $i := i+1$

4. $x^{(k+1,i-1)} = m(X^{(k+1,i-1)})$, (midpoint of $X^{(k+1,i-1)}$)

5. $$X^{(k+1,i)} = \left\{ x^{(k+1,i-1)} - \frac{1}{f'\left(x^{(k+1,i+1)}\right)} \left[f(x^{(k+1,i-1)}) + \sum_{\upsilon=2}^{i} \frac{1}{\upsilon!} f^{\upsilon}(x^{(k+1,i-1)})(X^{(k+1,i-1)} - x^{(k+1,i-1)})^{\upsilon} + \frac{1}{(i+1)!} F_{i+1}(X^{(k+1,i-1)} - x^{(k+1,i-1)})^{(i+1)} \right] \right\} \cap X^{(k+1,i-1)}$$

6. If $i < p$ proceed to step 3

7. If $i := p$ and $\left\| x^{(k+1)} \right\| < \varepsilon$, proceed to step 9

8. $k := k+1$, proceed to step 2

9. $X^{(k+1)} = X^{(k+1,p)}$

10. Stop.

NUMERICAL EXAMPLE

The following results are obtained by using FORTRAN 95 with p=5.

Example 1

1. Function $f(x) = (x-1)(x^4+1)$ which has $\xi = 1$ in the $X^{(0)} = [0.8, 2.0]$. By using algorithm EHR, table 1 and algorithm MEHR, table 2.

TABLE 1: Result for example 1 for EHR method

k	i	$X^{(k+1,i)}$	Width
0	0	[0.8 , 1.360476734693878]	5.604767346938775E-1
	1	[0.8 , 1.190008722564928]	3.900087225649278E-1
	2	[0.8 , 1.190008722564928]	3.900087225649278E-1
	3	[0.8 , 1.190008722564928]	3.900087225649278E-1
	4	[0.8 , 1.190008722564928]	3.900087225649278E-1
	5	[0.8 , 1.190008722564928]	3.900087225649278E-1
1	0	[0.9952062428612648,1.004896558643708]	9.690315782443393E-3
	1	[0.9972549737651973,1.000050132995012]	2.795159229814215E-3
	2	[0.9999972767236106,1.000040037065030]	4.276034141892371E-5
	3	[0.9999991880464568,1.000000055189234]	8.671427772499740E-7
	4	[0.9999999988875318,1.000000016365388]	1.747785638439581E-8
	5	[0.9999999999701185,1.000000000022424]	3.523057401366714E-10
2	0	[0.9999999998525461,1.000000000022424]	1.698782226000617E-10
	1	[1.0 , 1.0]	0

CPU time: 6.000000000495E-2 second. Ends for $k = 2$, $i = 1$.

TABLE 2: Result for example 1 for MEHR method

k	i	$X^{(k+1,i)}$	Width
0	0	[0.8 , 1.360476734693878]	5.604767346938775E-1
	1	[0.8 , 1.360476734693878]	5.604767346938775E-1
	2	[0.8 , 1.360476734693878]	5.604767346938775E-1
	3	[0.8 , 1.249357699095451]	4.493576990954507E-1
	4	[0.8613848269081681,1.140988617066789]	2.796037901586213E-1
	5	[0.9519185792678505,1.048087047267594]	9.616846799974310E-2
1	0	[0.9999971867006199,1.000002698439822]	5.511739202157528E-6
	1	[0.9999999997873517,1.000000000212661]	4.253097873885281E-10
	2	[1.0 , 1.0]	0

CPU Time: 1.0000 E^{-16} Second. Ends for k = 1, i = 2.

The following results are obtained by using FORTRAN 95 with p=5.

Example 2

1. $f(x) = x^7 + 3x^6 - 4x^5 - 12x^4 - x^3 - 3x^2 + 4x + 12$ (Alefeld G and Herzberger J [3]) is a function with $\xi = 2 \in X^{(0)}$= [1.8, 2.4]. By using algorithm EHR, table 3 and algorithm MEHR1 table 4.

TABLE 3 : Result for example 2 for EHR Method

k	i	$X^{(k+1,i)}$	Width
0	0	[1.8 , 2.072761807784166]	2.727618077841656E-1
	1	[1.8 , 2.019056626446785]	2.190566264467846E-1
	2	[1.826122979011974,2.019056626446785]	1.929336474348109E-1
	3	[1.850956931473188,2.019056626446785]	1.681005949735963E-1
	4	[1.883211195891202,2.019056626446785]	1.358454305555825E-1
	5	[1.917209908731083,2.019056626446785]	1.018467177157016E-1
1	0	[1.974345582482652,2.019056626446785]	4.471104396413250E-2
	1	[1.980491752716859,2.002855856641086]	2.236410392422750E-2
	2	[1.999294933278730,2.002488029279333]	3.193096000603246E-3
	3	[1.999516485730370,2.000129998588562]	6.135128581927773E-4
	4	[1.999975755335749,2.000089245487137]	1.134901513888398E-4
	5	[1.999983371164678,2.000004509003628]	2.113783895008048E-5
2	0	[1.999995223754924,2.000004509003628]	9.285248703916338E-6
	1	[1.999999999280437,2.000000000092703]	8.12260322129605E-10
	2	[1.99999999999996,2.0000000000000023]	2.620126338115369E-14
	3	[2.0 , 2.0]	0

CPU time: 5.999999999767E-2 second. Ends for: k = 2, i= 3.

TABLE 4: Result for example 2 for MEHR method

k	i	$X^{(k+1,i)}$	Width
0	0	[1.8 , 2.072761807784166]	2.727618077841656E-1
	1	[1.8 , 2.072761807784166]	2.727618077841656E-1
	2	[1.935577351822619,2.0727618077841666]	1.371844559615469E-1
	3	[1.987115122108837,2.01297353957179666]	2.585841746295903E-2
	4	[1.999562526067573,2.0004374840846666]	8.749580170925864E-4
	5	[1.999999505362727,2.0000004946372730]	9.892745451711704E-7
1	0	[2.0 , 2.0]	0

CPU time: 5.000000000291E-2 Second. Ends for $k = 1$, $i = 0$.

CONCLUSION

The numerical results above clearly shown that the MEHR procedure gives better results in term of processing time and the width of the last inclusion of the interval.

In fact, the R-order of convergence of MEHR is at least $p+3$,where

$$O_R\left(MEHR,\xi \right) \geq p+3,\left(p \geq 1\right)$$

In the other hand, the R-order of convergence of EHR method is at least p + *1,* that is

$$O_R\left(EHR,\xi \right) \geq p+1,\left(p \geq 1 \right).$$

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